4.9 NONLINEAR DIFFERENTIAL EQUATIONS

REVIEW MATERIAL

- Sections 2.2 and 2.5
- Section 4.2
- A review of Taylor series from calculus is also recommended.

INTRODUCTION The difficulties that surround higher-order *nonlinear* differential equations and the few methods that yield analytic solutions are examined next. Two of the solution methods considered in this section employ a change of variable to reduce a second-order DE to a first-order DE. In that sense these methods are analogous to the material in Section 4.2.

SOME DIFFERENCES There are several significant differences between linear and nonlinear differential equations. We saw in Section 4.1 that homogeneous linear equations of order two or higher have the property that a linear combination of solutions is also a solution (Theorem 4.1.2). Nonlinear equations do not possess this property of superposability. See Problems 1 and 18 in Exercises 4.9. We can find general solutions of linear first-order DEs and higher-order equations with constant coefficients. Even when we can solve a nonlinear first-order differential equation in the form of a one-parameter family, this family does not, as a rule, represent a general solution. Stated another way, nonlinear first-order DEs can possess singular solutions, whereas linear equations cannot. But the major difference between linear and nonlinear equations of order two or higher lies in the realm of solvability. Given a linear equation, there is a chance that we can find some form of a solution that we can look at—an explicit solution or perhaps a solution in the form of an infinite series (see Chapter 6). On the other hand, nonlinear higher-order differential equations virtually defy solution by analytical methods. Although this might sound disheartening, there are still things that can be done. As was pointed out at the end of Section 1.3, we can always analyze a nonlinear DE qualitatively and numerically.

Let us make it clear at the outset that nonlinear higher-order differential equations are important—dare we say even more important than linear equations?—because as we fine-tune the mathematical model of, say, a physical system, we also increase the likelihood that this higher-resolution model will be nonlinear.

We begin by illustrating an analytical method that *occasionally* enables us to find explicit/implicit solutions of special kinds of nonlinear second-order differential equations.

REDUCTION OF ORDER Nonlinear second-order differential equations $F(x, y', y'') = 0$, where the dependent variable y is missing, and $F(y, y', y'') = 0$, where the independent variable x is missing, can sometimes be solved by using first-order methods. Each equation can be reduced to a first-order equation by means of the substitution $u = y'$.

The next example illustrates the substitution technique for an equation of the form $F(x, y', y'') = 0$. If $u = y'$, then the differential equation becomes $F(x, u, u') = 0$. If we can solve this last equation for u, we can find y by integration. Note that since we are solving a second-order equation, its solution will contain two arbitrary constants.

EXAMPLE 1 **Dependent Variable y Is Missing**

Solve $y'' = 2x(y')^2$.

SOLUTION If we let $u = y'$, then $du/dx = y''$. After substituting, the second-order equation reduces to a first-order equation with separable variables; the independent variable is x and the dependent variable is u:

$$\frac{du}{dx} = 2xu^2 \quad \text{or} \quad \frac{du}{u^2} = 2x\,dx$$

$$\int u^{-2}\,du = \int 2x\,dx$$

$$-u^{-1} = x^2 + c_1^2.$$

The constant of integration is written as c_1^2 for convenience. The reason should be obvious in the next few steps. Because $u^{-1} = 1/y'$, it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2},$$

and so $$y = -\int \frac{dx}{x^2 + c_1^2} \quad \text{or} \quad y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2. \qquad \blacksquare$$

Next we show how to solve an equation that has the form $F(y, y', y'') = 0$. Once more we let $u = y'$, but because the independent variable x is missing, we use this substitution to transform the differential equation into one in which the independent variable is y and the dependent variable is u. To this end we use the Chain Rule to compute the second derivative of y:

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}.$$

In this case the first-order equation that we must now solve is

$$F\left(y, u, u\frac{du}{dy}\right) = 0.$$

EXAMPLE 2 Independent Variable x Is Missing

Solve $yy'' = (y')^2$.

SOLUTION With the aid of $u = y'$, the Chain Rule shown above, and separation of variables, the given differential equation becomes

$$y\left(u\frac{du}{dy}\right) = u^2 \quad \text{or} \quad \frac{du}{u} = \frac{dy}{y}.$$

Integrating the last equation then yields $\ln|u| = \ln|y| + c_1$, which, in turn, gives $u = c_2 y$, where the constant $\pm e^{c_1}$ has been relabeled as c_2. We now resubstitute $u = dy/dx$, separate variables once again, integrate, and relabel constants a second time:

$$\int \frac{dy}{y} = c_2 \int dx \quad \text{or} \quad \ln|y| = c_2 x + c_3 \quad \text{or} \quad y = c_4 e^{c_2 x}. \qquad \blacksquare$$

USE OF TAYLOR SERIES In some instances a solution of a nonlinear initial-value problem, in which the initial conditions are specified at x_0, can be approximated by a Taylor series centered at x_0.

EXAMPLE 3 Taylor Series Solution of an IVP

Let us assume that a solution of the initial-value problem

$$y'' = x + y - y^2, \quad y(0) = -1, \quad y'(0) = 1 \tag{1}$$

exists. If we further assume that the solution $y(x)$ of the problem is analytic at 0, then $y(x)$ possesses a Taylor series expansion centered at 0:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots. \tag{2}$$

Note that the values of the first and second terms in the series (2) are known since those values are the specified initial conditions $y(0) = -1$, $y'(0) = 1$. Moreover, the differential equation itself defines the value of the second derivative at 0: $y''(0) = 0 + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2$. We can then find expressions for the higher derivatives y''', $y^{(4)}, \ldots$ by calculating the successive derivatives of the differential equation:

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy' \tag{3}$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy') = y'' - 2yy'' - 2(y')^2 \tag{4}$$

$$y^{(5)}(x) = \frac{d}{dx}(y'' - 2yy'' - 2(y')^2) = y''' - 2yy''' - 6y'y'', \tag{5}$$

and so on. Now using $y(0) = -1$ and $y'(0) = 1$, we find from (3) that $y'''(0) = 4$. From the values $y(0) = -1$, $y'(0) = 1$, and $y''(0) = -2$ we find $y^{(4)}(0) = -8$ from (4). With the additional information that $y'''(0) = 4$, we then see from (5) that $y^{(5)}(0) = 24$. Hence from (2) the first six terms of a series solution of the initial-value problem (1) are

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots. \qquad \blacksquare$$

USE OF A NUMERICAL SOLVER Numerical methods, such as Euler's method or the Runge-Kutta method, are developed solely for first-order differential equations and then are extended to systems of first-order equations. To analyze an nth-order initial-value problem numerically, we express the nth-order ODE as a system of n first-order equations. In brief, here is how it is done for a second-order initial-value problem: First, solve for y''—that is, put the DE into normal form $y'' = f(x, y, y')$—and then let $y' = u$. For example, if we substitute $y' = u$ in

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y(x_0) = y_0, \quad y'(x_0) = u_0, \tag{6}$$

then $y'' = u'$ and $y'(x_0) = u(x_0)$, so the initial-value problem (6) becomes

$$\textit{Solve:} \quad \begin{cases} y' = u \\ u' = f(x, y, u) \end{cases}$$

$$\textit{Subject to:} \quad y(x_0) = y_0,\ u(x_0) = u_0.$$

However, it should be noted that a commercial numerical solver *might not* require* that you supply the system.

*Some numerical solvers require only that a second-order differential equation be expressed in normal form $y'' = f(x, y, y')$. The translation of the single equation into a system of two equations is then built into the computer program, since the first equation of the system is always $y' = u$ and the second equation is $u' = f(x, y, u)$.

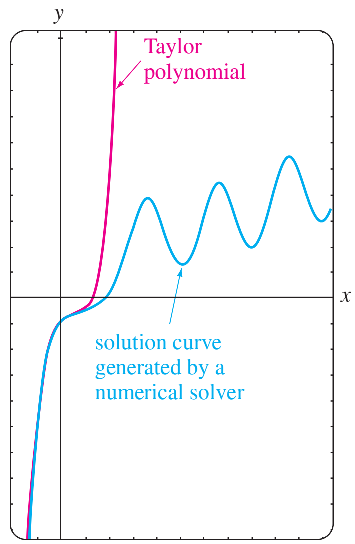


FIGURE 4.9.1 Comparison of two approximate solutions

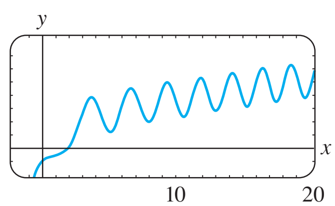


FIGURE 4.9.2 Numerical solution curve for the IVP in (1)

EXAMPLE 4 Graphical Analysis of Example 3

Following the foregoing procedure, we find that the second-order initial-value problem in Example 3 is equivalent to

$$\frac{dy}{dx} = u$$
$$\frac{du}{dx} = x + y - y^2$$

with initial conditions $y(0) = -1$, $u(0) = 1$. With the aid of a numerical solver we get the solution curve shown in blue in Figure 4.9.1. For comparison the graph of the fifth-degree Taylor polynomial $T_5(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5$ is shown in red. Although we do not know the interval of convergence of the Taylor series obtained in Example 3, the closeness of the two curves in a neighborhood of the origin suggests that the power series may converge on the interval $(-1, 1)$. ■

QUALITATIVE QUESTIONS The blue graph in Figure 4.9.1 raises some questions of a qualitative nature: Is the solution of the original initial-value problem oscillatory as $x \to \infty$? The graph generated by a numerical solver on the larger interval shown in Figure 4.9.2 would seem to *suggest* that the answer is yes. But this single example—or even an assortment of examples—does not answer the basic question as to whether *all* solutions of the differential equation $y'' = x + y - y^2$ are oscillatory in nature. Also, what is happening to the solution curve in Figure 4.9.2 when x is near -1? What is the behavior of solutions of the differential equation as $x \to -\infty$? Are solutions bounded as $x \to \infty$? Questions such as these are not easily answered, in general, for nonlinear second-order differential equations. But certain kinds of second-order equations lend themselves to a systematic qualitative analysis, and these, like their first-order relatives encountered in Section 2.1, are the kind that have no explicit dependence on the independent variable. Second-order ODEs of the form

$$F(y, y', y'') = 0 \qquad \text{or} \qquad \frac{d^2y}{dx^2} = f(y, y'),$$

equations free of the independent variable x, are called **autonomous.** The differential equation in Example 2 is autonomous, and because of the presence of the x term on its right-hand side, the equation in Example 3 is nonautonomous. For an in-depth treatment of the topic of stability of autonomous second-order differential equations and autonomous systems of differential equations, refer to Chapter 10 in *Differential Equations with Boundary-Value Problems.*

EXERCISES 4.9

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems 1 and 2 verify that y_1 and y_2 are solutions of the given differential equation but that $y = c_1y_1 + c_2y_2$ is, in general, not a solution.

1. $(y'')^2 = y^2$; $\quad y_1 = e^x, y_2 = \cos x$

2. $yy'' = \frac{1}{2}(y')^2$; $\quad y_1 = 1, y_2 = x^2$

In Problems 3–8 solve the given differential equation by using the substitution $u = y'$.

3. $y'' + (y')^2 + 1 = 0$ **4.** $y'' = 1 + (y')^2$

5. $x^2y'' + (y')^2 = 0$ **6.** $(y + 1)y'' = (y')^2$

7. $y'' + 2y(y')^3 = 0$ **8.** $y^2y'' = y'$

9. Consider the initial-value problem

$$y'' + yy' = 0, \quad y(0) = 1, y'(0) = -1.$$

(a) Use the DE and a numerical solver to graph the solution curve.

(b) Find an explicit solution of the IVP. Use a graphing utility to graph this solution.

(c) Find an interval of definition for the solution in part (b).

10. Find two solutions of the initial-value problem

$$(y'')^2 + (y')^2 = 1, \quad y\left(\frac{\pi}{2}\right) = \frac{1}{2}, \quad y'\left(\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}.$$

Use a numerical solver to graph the solution curves.

In Problems 11 and 12 show that the substitution $u = y'$ leads to a Bernoulli equation. Solve this equation (see Section 2.5).

11. $xy'' = y' + (y')^3$ **12.** $xy'' = y' + x(y')^2$

In Problems 13–16 proceed as in Example 3 and obtain the first six nonzero terms of a Taylor series solution, centered at 0, of the given initial-value problem. Use a numerical solver and a graphing utility to compare the solution curve with the graph of the Taylor polynomial.

13. $y'' = x + y^2, \quad y(0) = 1, y'(0) = 1$

14. $y'' + y^2 = 1, \quad y(0) = 2, y'(0) = 3$

15. $y'' = x^2 + y^2 - 2y', \quad y(0) = 1, y'(0) = 1$

16. $y'' = e^y, \quad y(0) = 0, y'(0) = -1$

17. In calculus the curvature of a curve that is defined by a function $y = f(x)$ is defined as

$$\kappa = \frac{y''}{[1 + (y')^2]^{3/2}}.$$

Find $y = f(x)$ for which $\kappa = 1$. [*Hint*: For simplicity, ignore constants of integration.]

Discussion Problems

18. In Problem 1 we saw that $\cos x$ and e^x were solutions of the nonlinear equation $(y'')^2 - y^2 = 0$. Verify that $\sin x$ and e^{-x} are also solutions. Without attempting to solve the differential equation, discuss how these explicit solutions can be found by using knowledge about linear equations. Without attempting to verify, discuss why the linear combinations $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$ and $y = c_2e^{-x} + c_4 \sin x$ are not, in general, solutions, but the two special linear combinations $y = c_1e^x + c_2e^{-x}$ and $y = c_3 \cos x + c_4 \sin x$ *must* satisfy the differential equation.

19. Discuss how the method of reduction of order considered in this section can be applied to the third-order differential equation $y''' = \sqrt{1 + (y'')^2}$. Carry out your ideas and solve the equation.

20. Discuss how to find an alternative two-parameter family of solutions for the nonlinear differential equation $y'' = 2x(y')^2$ in Example 1. [*Hint*: Suppose that $-c_1^2$ is used as the constant of integration instead of $+c_1^2$.]

Mathematical Models

21. Motion in a Force Field A mathematical model for the position $x(t)$ of a body moving rectilinearly on the x-axis in an inverse-square force field is given by

$$\frac{d^2x}{dt^2} = -\frac{k^2}{x^2}.$$

Suppose that at $t = 0$ the body starts from rest from the position $x = x_0$, $x_0 > 0$. Show that the velocity of the body at time t is given by $v^2 = 2k^2(1/x - 1/x_0)$. Use the last expression and a CAS to carry out the integration to express time t in terms of x.

22. A mathematical model for the position $x(t)$ of a moving object is

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

Use a numerical solver to graphically investigate the solutions of the equation subject to $x(0) = 0$, $x'(0) = x_1$, $x_1 \geq 0$. Discuss the motion of the object for $t \geq 0$ and for various choices of x_1. Investigate the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \sin x = 0$$

in the same manner. Give a possible physical interpretation of the dx/dt term.

CHAPTER 4 IN REVIEW

Answers to selected odd-numbered problems begin on page ANS-6.

Answer Problems 1–4 without referring back to the text. Fill in the blank or answer true or false.

1. The only solution of the initial-value problem $y'' + x^2y = 0$, $y(0) = 0$, $y'(0) = 0$ is __________.

2. For the method of undetermined coefficients, the assumed form of the particular solution y_p for $y'' - y = 1 + e^x$ is __________.

3. A constant multiple of a solution of a linear differential equation is also a solution. __________

4. If the set consisting of two functions f_1 and f_2 is linearly independent on an interval I, then the Wronskian $W(f_1, f_2) \neq 0$ for all x in I. __________

5. Give an interval over which the set of two functions $f_1(x) = x^2$ and $f_2(x) = x|x|$ is linearly independent.

Then give an interval over which the set consisting of f_1 and f_2 is linearly dependent.

6. Without the aid of the Wronskian, determine whether the given set of functions is linearly independent or linearly dependent on the indicated interval.

(a) $f_1(x) = \ln x, f_2(x) = \ln x^2, (0, \infty)$

(b) $f_1(x) = x^n, f_2(x) = x^{n+1}, n = 1, 2, \ldots, (-\infty, \infty)$

(c) $f_1(x) = x, f_2(x) = x + 1, (-\infty, \infty)$

(d) $f_1(x) = \cos\left(x + \dfrac{\pi}{2}\right), f_2(x) = \sin x, (-\infty, \infty)$

(e) $f_1(x) = 0, f_2(x) = x, (-5, 5)$

(f) $f_1(x) = 2, f_2(x) = 2x, (-\infty, \infty)$

(g) $f_1(x) = x^2, f_2(x) = 1 - x^2, f_3(x) = 2 + x^2, (-\infty, \infty)$

(h) $f_1(x) = xe^{x+1}, f_2(x) = (4x - 5)e^x,$
$f_3(x) = xe^x, (-\infty, \infty)$

7. Suppose $m_1 = 3$, $m_2 = -5$, and $m_3 = 1$ are roots of multiplicity one, two, and three, respectively, of an auxiliary equation. Write down the general solution of the corresponding homogeneous linear DE if it is

(a) an equation with constant coefficients,

(b) a Cauchy-Euler equation.

8. Consider the differential equation $ay'' + by' + cy = g(x)$, where a, b, and c are constants. Choose the input functions $g(x)$ for which the method of undetermined coefficients is applicable and the input functions for which the method of variation of parameters is applicable.

(a) $g(x) = e^x \ln x$ **(b)** $g(x) = x^3 \cos x$

(c) $g(x) = \dfrac{\sin x}{e^x}$ **(d)** $g(x) = 2x^{-2}e^x$

(e) $g(x) = \sin^2 x$ **(f)** $g(x) = \dfrac{e^x}{\sin x}$

In Problems 9–24 use the procedures developed in this chapter to find the general solution of each differential equation.

9. $y'' - 2y' - 2y = 0$

10. $2y'' + 2y' + 3y = 0$

11. $y''' + 10y'' + 25y' = 0$

12. $2y''' + 9y'' + 12y' + 5y = 0$

13. $3y''' + 10y'' + 15y' + 4y = 0$

14. $2y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0$

15. $y'' - 3y' + 5y = 4x^3 - 2x$

16. $y'' - 2y' + y = x^2e^x$

17. $y''' - 5y'' + 6y' = 8 + 2 \sin x$

18. $y''' - y'' = 6$

19. $y'' - 2y' + 2y = e^x \tan x$

20. $y'' - y = \dfrac{2e^x}{e^x + e^{-x}}$

21. $6x^2y'' + 5xy' - y = 0$

22. $2x^3y''' + 19x^2y'' + 39xy' + 9y = 0$

23. $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

24. $x^2y'' - xy' + y = x^3$

25. Write down the form of the general solution $y = y_c + y_p$ of the given differential equation in the two cases $\omega \neq \alpha$ and $\omega = \alpha$. Do not determine the coefficients in y_p.

(a) $y'' + \omega^2 y = \sin \alpha x$ **(b)** $y'' - \omega^2 y = e^{\alpha x}$

26. **(a)** Given that $y = \sin x$ is a solution of

$$y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0,$$

find the general solution of the DE *without the aid of a calculator or a computer.*

(b) Find a linear second-order differential equation with constant coefficients for which $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the associated homogeneous equation and $y_p = \frac{1}{2}x^2 - x$ is a particular solution of the nonhomogeneous equation.

27. **(a)** Write the general solution of the fourth-order DE $y^{(4)} - 2y'' + y = 0$ entirely in terms of hyperbolic functions.

(b) Write down the form of a particular solution of $y^{(4)} - 2y'' + y = \sinh x$.

28. Consider the differential equation

$$x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3.$$

Verify that $y_1 = x$ is one solution of the associated homogeneous equation. Then show that the method of reduction of order discussed in Section 4.2 leads to a second solution y_2 of the homogeneous equation as well as a particular solution y_p of the nonhomogeneous equation. Form the general solution of the DE on the interval $(0, \infty)$.

In Problems 29–34 solve the given differential equation subject to the indicated conditions.

29. $y'' - 2y' + 2y = 0, \quad y\left(\dfrac{\pi}{2}\right) = 0, y(\pi) = -1$

30. $y'' + 2y' + y = 0, \quad y(-1) = 0, y'(0) = 0$

31. $y'' - y = x + \sin x, \quad y(0) = 2, y'(0) = 3$

32. $y'' + y = \sec^3 x, \quad y(0) = 1, y'(0) = \dfrac{1}{2}$

33. $y'y'' = 4x, \quad y(1) = 5, y'(1) = 2$

34. $2y'' = 3y^2, \quad y(0) = 1, y'(0) = 1$

35. **(a)** Use a CAS as an aid in finding the roots of the auxiliary equation for

$$12y^{(4)} + 64y''' + 59y'' - 23y' - 12y = 0.$$

Give the general solution of the equation.

(b) Solve the DE in part (a) subject to the initial conditions $y(0) = -1$, $y'(0) = 2$, $y''(0) = 5$, $y'''(0) = 0$. Use a CAS as an aid in solving the resulting systems of four equations in four unknowns.

36. Find a member of the family of solutions of $xy'' + y' + \sqrt{x} = 0$ whose graph is tangent to the x-axis at $x = 1$. Use a graphing utility to graph the solution curve.

In Problems 37–40 use systematic elimination to solve the given system.

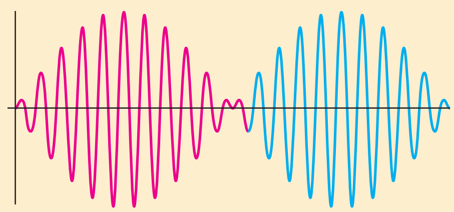
37.
$$\begin{aligned} \frac{dx}{dt} + \frac{dy}{dt} &= 2x + 2y + 1 \\ \frac{dx}{dt} + 2\frac{dy}{dt} &= y + 3 \end{aligned}$$

38.
$$\begin{aligned} \frac{dx}{dt} &= 2x + y + t - 2 \\ \frac{dy}{dt} &= 3x + 4y - 4t \end{aligned}$$

39.
$$\begin{aligned} (D - 2)x \qquad\quad - y &= -e^t \\ -3x + (D - 4)y &= -7e^t \end{aligned}$$

40.
$$\begin{aligned} (D + 2)x + (D + 1)y &= \sin 2t \\ 5x + (D + 3)y &= \cos 2t \end{aligned}$$

5 MODELING WITH HIGHER-ORDER DIFFERENTIAL EQUATIONS



We have seen that a single differential equation can serve as a mathematical model for diverse physical systems. For this reason we examine just one application, the motion of a mass attached to a spring, in great detail in Section 5.1. Except for terminology and physical interpretations of the four terms in the linear equation $ay'' + by' + cy = g(t)$, the mathematics of, say, an electrical series circuit is identical to that of vibrating spring/mass system. Forms of this linear second-order DE appear in the analysis of problems in many diverse areas of science and engineering. In Section 5.1 we deal exclusively with initial-value problems, whereas in Section 5.2 we examine applications described by boundary-value problems. In Section 5.2 we also see how some boundary-value problems lead to the important concepts of *eigenvalues* and *eigenfunctions*. Section 5.3 begins with a discussion on the differences between linear and nonlinear springs; we then show how the simple pendulum and a suspended wire lead to nonlinear models.

5.1 LINEAR MODELS: INITIAL-VALUE PROBLEMS

REVIEW MATERIAL

- Sections 4.1, 4.3, and 4.4
- Problems 29–36 in Exercises 4.3
- Problems 27–36 in Exercises 4.4

INTRODUCTION In this section we are going to consider several linear dynamical systems in which each mathematical model is a second-order differential equation with constant coefficients along with initial conditions specified at a time that we shall take to be $t = 0$:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Recall that the function g is the **input, driving function,** or **forcing function** of the system. A solution $y(t)$ of the differential equation on an interval I containing $t = 0$ that satisfies the initial conditions is called the **output** or **response** of the system.

5.1.1 SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

HOOKE'S LAW Suppose that a flexible spring is suspended vertically from a rigid support and then a mass m is attached to its free end. The amount of stretch, or elongation, of the spring will of course depend on the mass; masses with different weights stretch the spring by differing amounts. By Hooke's law the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. Simply stated, $F = ks$, where k is a constant of proportionality called the **spring constant.** The spring is essentially characterized by the number k. For example, if a mass weighing 10 pounds stretches a spring $\frac{1}{2}$ foot, then $10 = k\left(\frac{1}{2}\right)$ implies $k = 20$ lb/ft. Necessarily then, a mass weighing, say, 8 pounds stretches the same spring only $\frac{2}{5}$ foot.

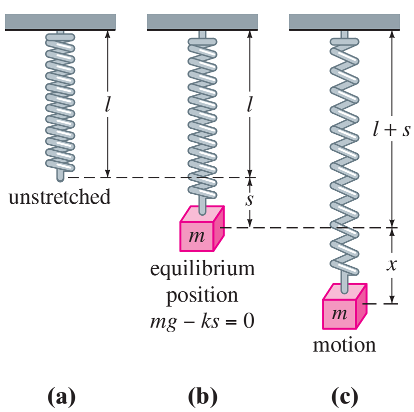


FIGURE 5.1.1 Spring/mass system

NEWTON'S SECOND LAW After a mass m is attached to a spring, it stretches the spring by an amount s and attains a position of equilibrium at which its weight W is balanced by the restoring force ks. Recall that weight is defined by $W = mg$, where mass is measured in slugs, kilograms, or grams and $g = 32$ ft/s^2, 9.8 m/s^2, or 980 cm/s^2, respectively. As indicated in Figure 5.1.1(b), the condition of equilibrium is $mg = ks$ or $mg - ks = 0$. If the mass is displaced by an amount x from its equilibrium position, the restoring force of the spring is then $k(x + s)$. Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces—**free motion**—we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + \underbrace{mg - ks}_{\text{zero}} = -kx. \tag{1}$$

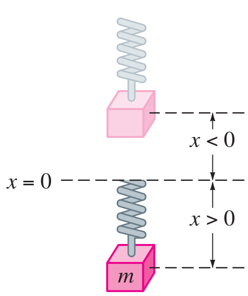


FIGURE 5.1.2 Direction below the equilibrium position is positive.

The negative sign in (1) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we adopt the convention that displacements measured *below* the equilibrium position are positive. See Figure 5.1.2.